25. Fourier Transform: Concept & Equations

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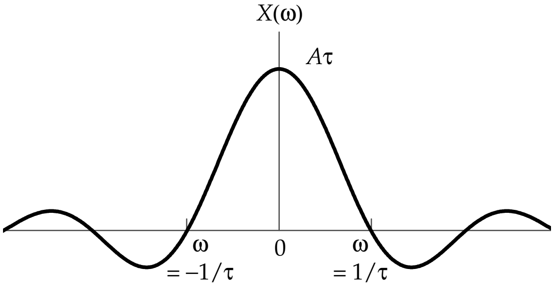


Fourier series provides **frequency-domain** representation of periodic signal $x(t)$, which shows the **frequency content** of that signal (i.e., its harmonic content).

A lot of information can be gained from the frequency-domain representation of $x(t)$, such as signal bandwidth, how the signal interacts with hardware systems (such as filters), what are the signal properties, how to perform signal conditioning, how the signal interacts with antennas, etc.

But **Fourier series** works only for periodic signals, and is not applicable for aperiodic signals $x(t)$? That is why we use the more general (yet similar) **Fourier transform**, which works for both periodic and aperiodic signals.

Fourier transform, denote by $X(\omega)$, of the signal $x(t)$, periodic or aperiodic, is defined by the integral,

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$X(\omega)$ is, in general, complex with **magnitude** $|X(\omega)|$ and **phase** $\angle X(\omega)$, and it is a function of the continuous angular frequency ω. If Fourier transform $X(\omega)$ was given, you can figure out the original signal $x(t)$ using the inverse operation, called inverse Fourier transform, given by the integral,

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

To visualize the Fourier transform $X(\omega)$, we typically draw the **magnitude** $|X(\omega)|$ versus frequency ω. This is called the **magnitude spectrum density** of $x(t)$, and it is always two-sided. Notice also that in this case we use continuous-frequency ω (which is general enough to include both smooth curves and impulses), unlike the Fourier series spectrum, which has discrete-frequency $n\omega_0$, that can only support discrete lines.

We also plot the **phase** $\angle X(\omega)$ versus continuous frequency ω, known as **phase spectrum density** (also two-sided).

Similar to $\alpha_{-n} = \alpha_n^*$, for real-value $x(t)$, we have $X(\omega) = X^*(-\omega)$, which means the magnitude spectrum density $|X(\omega)|$ has **even symmetry**, while the phase spectrum density $\angle X(\omega)$ has **odd symmetry**.

Keep in mind:

Many times, we can combine the two plots (magnitude spectrum density $|X(\omega)|$ and phase spectrum density $\angle X(\omega)$) into one diagram $X(\omega)$, sometimes called amplitude spectrum density.

Unlike α_0, which represents the DC (or average value of $x(t)$), the value of $X(\omega)$ at $\omega = 0$ (i.e., $X(0)$) is **not** the DC or average value of $x(t)$; rather the DC value shows up in $X(\omega)$ as the area of the impulse at $\omega = 0$ (if such impulse exists). No impulse means the DC value is zero.

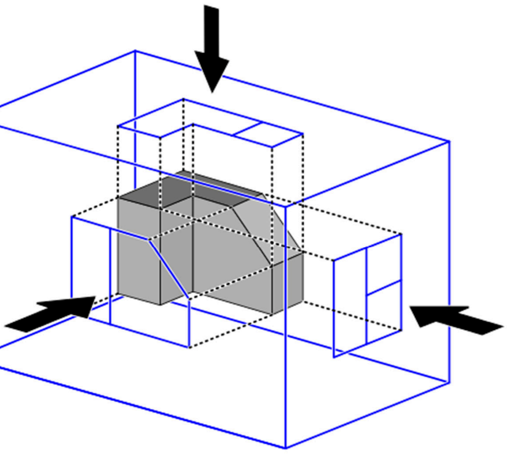
There is no such thing as negative frequency in real-life. The two-sided spectral densities are just a mathematical convenience (complex numbers help us out).

The two plots together (magnitude spectrum density and phase spectrum density) are the **frequency-domain** representation of $x(t)$. They show the **frequency content distribution** (or density) for $x(t)$. They represent an alternative (but equivalent) way to look at the signal in contrast to its **time-domain** representation. We say the signal was transformed from t-domain to f-domain using $\mathcal{F}\{\quad\}$ (and the opposite way using $\mathcal{F}^{-1}$), but actually it is just another view of the same signal.

$X(\omega)$ represents a **unique** description of the signal $x(t)$, hence knowing the spectral densities we can reconstruct the signal $x(t)$ [using $\mathcal{F}^{-1}$].

We say that $x(t)$ and $X(\omega)$ represent a Fourier transform pair,

$$x(t) \Leftrightarrow X(\omega)$$



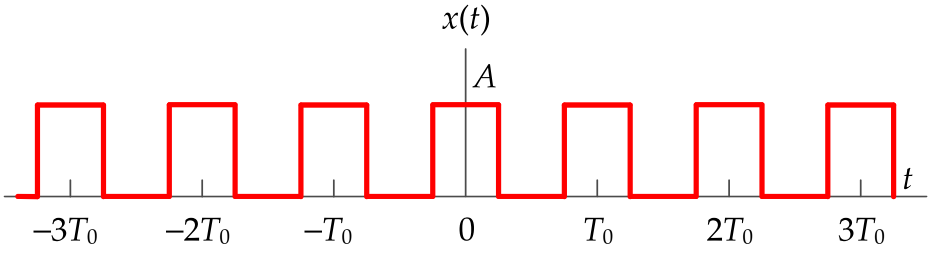
The Fourier transform $X(\omega)$ of **periodic** signal $x(t)$ is almost identical to the Fourier series complex exponential form α_n of that same periodic signal $x(t)$ (remember that both are complex values).

It can be easily proven that, for **periodic** signals, we can convert the Fourier series (complex exponential form) α_n to Fourier transform $X(\omega)$ by introducing only two modifications:

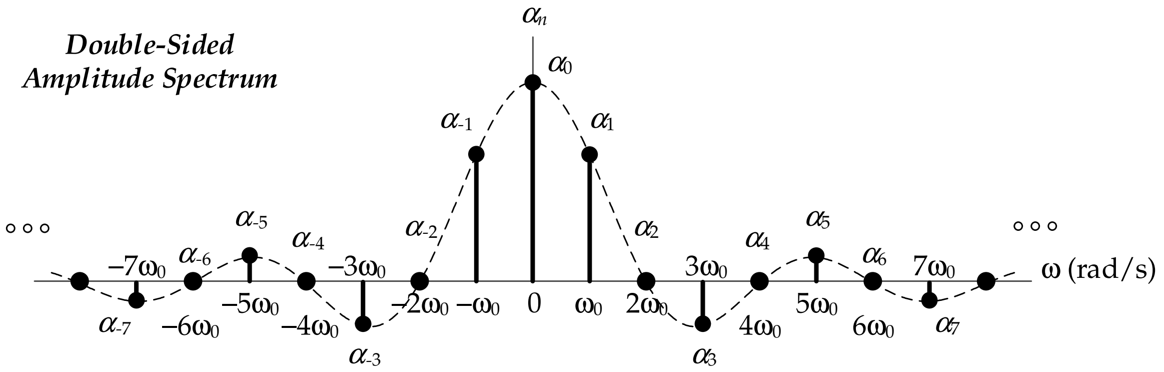
(1) Multiply each α_n value by a factor of 2π.

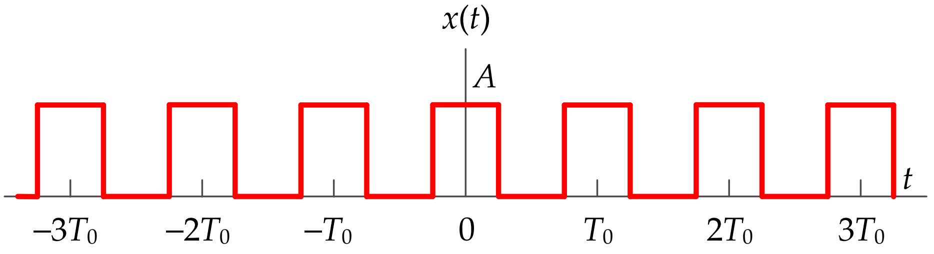
(2) Convert each discrete value α_n (line) in the spectrum into an impulse (with area of $2\pi\alpha_n$) in the spectrum density.

The proof is quite simple using Fourier transform properties, so we keep it for later. Rather, let us see an example.

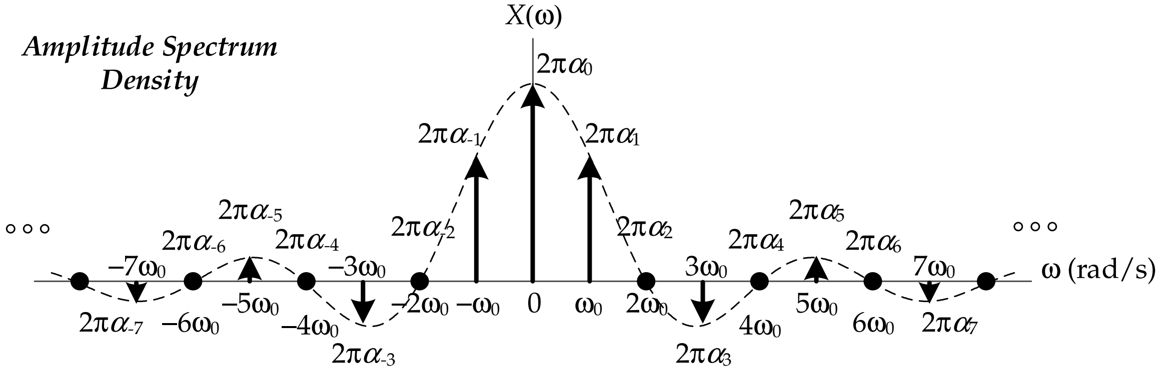


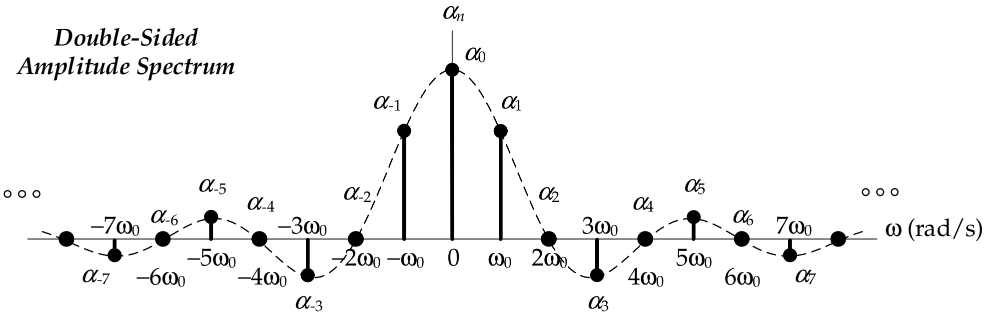
Double-Sided Amplitude Spectrum

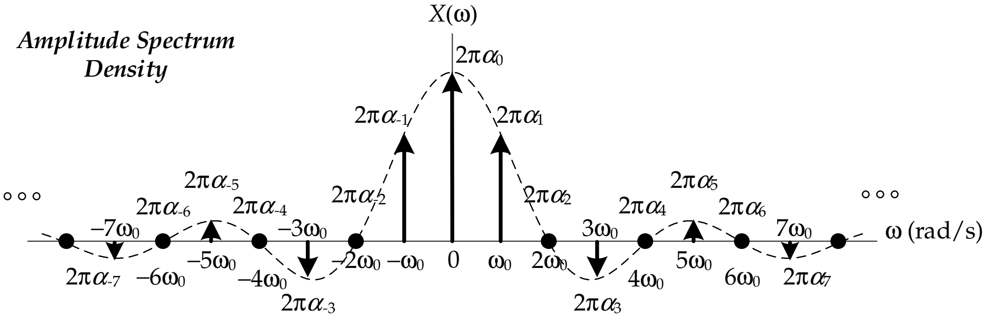




Amplitude Spectrum Density







This is due to the fact that periodic signals are actually the sum of an infinite number of sinusoids, and each sinusoid is equivalent to two impulses in Fourier transform (see examples *later*).

Conclusion:

For periodic signal $x(t)$, just find the corresponding α_n values using complex exponential Fourier series.

If you have trigonometric or compact Fourier series forms, then convert them to the complex exponential form.

Then modify such α_n values to get the corresponding Fourier transform $X(\omega)$.

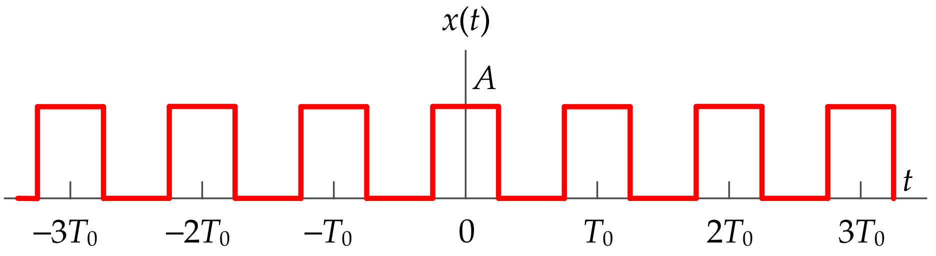
The Fourier transform $X(\omega)$ of the **aperiodic** signal $x(t)$ looks similar to the envelope (continuous-frequency smooth curve) of the Fourier series complex exponential form α_n of the corresponding periodic signal $\text{rep}_{T_0}\{x(t)\}$ (discrete-frequency result).

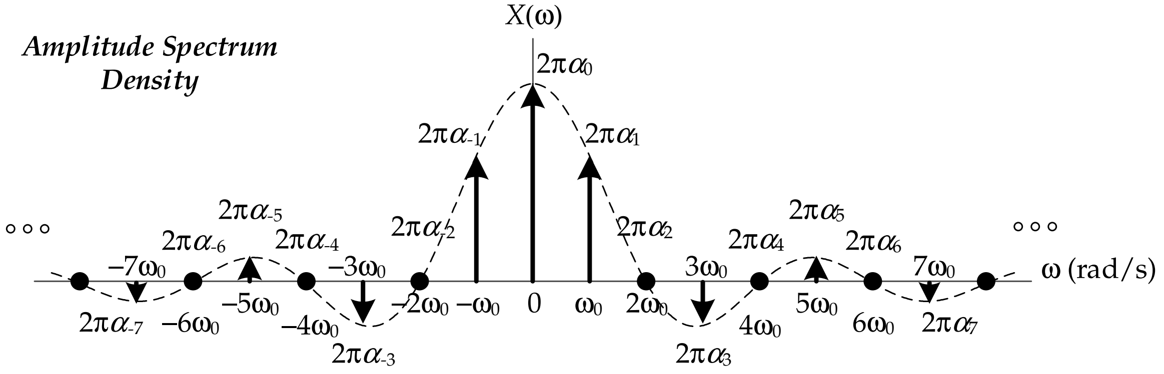
This is clear from the definition of both quantities (both are complex),

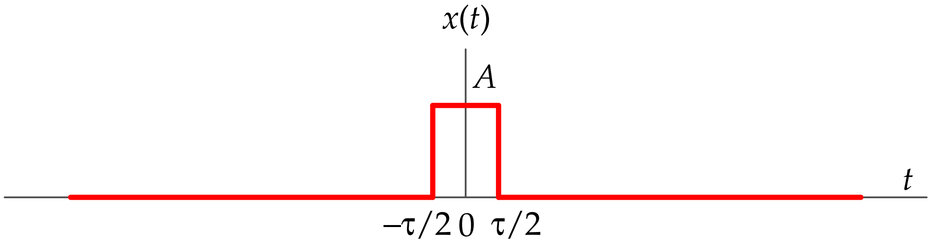
$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\alpha_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt \quad , \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

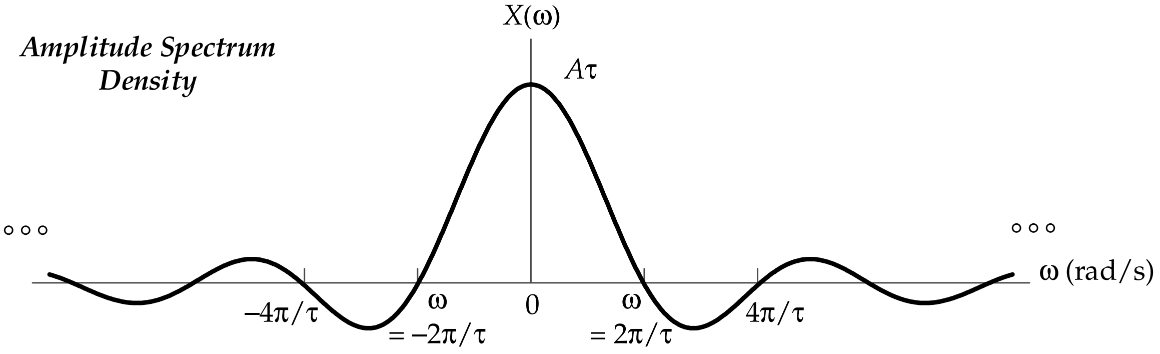
Notice the extra T_0 (fundamental period) and the conversion of the continuous-frequency ω into discrete-frequency $n\omega_0$ for periodic signal.







Amplitude Spectrum Density



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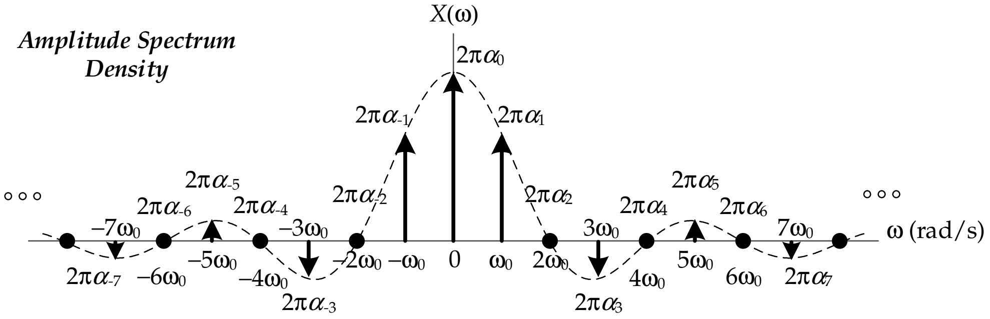

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